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Raising and lowering operators and their factorization for generalized orthogonal polynomials of hypergeometric type on homogeneous and non-homogeneous lattices

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Abstract
We complete the construction of raising and lowering operators, given in a previous work, for the orthogonal polynomials of hypergeometric type on a non-homogeneous lattice. We extend these operators to the generalized orthogonal polynomials, namely, those difference orthogonal polynomials that satisfy a similar difference equation of hypergeometric type.

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1. Introduction

Recently, we have presented a paper on the raising and lowering operators for the orthogonal polynomials (OPs) of hypergeometric type [1] (in connection with the factorization method defined by Hull and Infeld). In that paper, we covered only OPs of continuous and discrete variables on a uniform lattice, as well as orthonormal functions of continuous and discrete variables.

In this paper, we continue the construction of raising and lowering operators for OPs on a non-homogeneous lattice. The starting point is also the Rodriguez formula and the fundamental properties of OPs given by Nikiforov and collaborators [2, 3], which include the q-analogue of classical OPs of a discrete variable.

Atakishiyev and Suslov [4, 5] extended the classification of Nikiforov to OPs of a discrete variable defined by Andrews and Askey and proved that they satisfy a difference equation only in the cases of $x(s)$ linear, quadratic q-linear and q-quadratic.

The construction of raising and lowering operators on a non-uniform lattice was also worked out by Alvarez-Nodarse and Costas-Santos [6] and Alvarez-Nodarse and Arvesú [7] for the lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$.

Similar work has been carried out by Smirnov for OPs of hypergeometric type on homogeneous [8] and non-homogeneous [9] lattices, although the raising and lowering operators have been defined with respect to two indices: n, the order of polynomials, and m, the order of the difference derivatives of polynomials. In this paper, we define the raising and lowering operators with respect to one index only, either n or m.

In order to complete the classification of the OPs of hypergeometric type, we include the generalized classical OPs that also satisfy a difference/differential equation of hypergeometric type.

Since all classical OPs of a discrete variable lead in the limit to the corresponding OPs of a continuous variable, we begin in section 2 with the raising and lowering operators for generalized OPs of a continuous variable with respect to the index n, using the Rodriguez formula. In section 3 we repeat the same construction for generalized classical OPs of a discrete variable on a homogeneous lattice. In section 4 we extend the construction to classical OPs of a discrete variable on a non-homogeneous lattice in the general case, when $x(s) = c_1 s^2 + c_2 s + c_3$ or $x(s) = c_1 q^s + c_2 q^{-s} + c_3$.

In section 5 we complete the picture with the construction of raising and lowering operators for generalized classical OPs on non-homogeneous lattices, that include the q-analogue of classical OPs of a discrete variable. In all these cases, the raising and lowering operators are given with respect to one index, say, n. But the same operator can be considered, written in an appropriate form, for the raising and lowering operators with respect to index m.

2. Raising and lowering operators for generalized classical OPs of a continuous variable

Let $y_n(x)$ be an OP of a continuous variable satisfying the differential equation [2]

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) + \lambda_n y_n(x) = 0 \tag{1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degrees, respectively, and

$$\lambda_n = -n\left(\tau' + \tfrac{1}{2}(n-1)\sigma''\right). \tag{2}$$

It can be proven that the derivatives of $y_n(x)$, namely, $y_n^{(m)}(x) = v_{mn}(x)$, $m = 0, 1, \ldots, n-1$, satisfy a similar equation

$$\sigma(x)v_{mn}''(x) + \tau_m v_{mn}'(x) + \mu_{mn} v_{mn}(x) = 0 \tag{3}$$

with $\tau_m = \tau(x) + m\sigma'(x)$ and $\mu_{mn} = -(n-m)\left(\tau' + \frac{n+m-1}{2}\sigma''\right)$, $m = 0, 1, \ldots, n-1$.

We call these polynomials generalized OPs of hypergeometric type, some particular examples of which are the Legendre, Laguerre, Hermite and Jacobi generalized OPs[1].

The polynomials of hypergeometric type satisfy an orthognality property with respect to the weight function $\rho(x)$

$$\int_a^b y_\ell(x)y_n(x)\rho(x)\,\mathrm{d}x = \delta_{\ell n} d_n^2. \tag{4}$$

Similarly, the generalized OPs satisfy

$$\int_a^b v_{m_\ell}(x)v_{mn}(x)\rho_m(x)\,\mathrm{d}x = \delta_{\ell n} d_{mn}^2 \tag{5}$$

where d_n^2 and d_{mn}^2 are normalization constants.

It can be proven that [11]

$$d_{mn}^2 = d_{nn}^2\left(\prod_{k=m}^{n-1}\mu_{kn}\right)^{-1} \qquad d_{0n}^2 = d_{nn}^2\left(\prod_{k=0}^{n-1}\mu_{kn}\right)^{-1}$$

[1] Although some authors call these polynomials associated OPs, we prefer to call them generalized OPs in order to distinguish from the traditional name of associated classical OPs; see [10].

from which d_{nn}^2 can be eliminated. Therefore

$$d_{mn}^2 = d_{0n}^2 \prod_{k=0}^{m-1} \mu_{kn} \tag{6}$$

where d_{0n}^2 and d_n^2 are given in the tables of Nikiforov *et al* [12].

The generalized OPs of hypergeometric type can be calculated from the weight function $\rho_m(x) = \sigma(x)^m \rho(x)$, with the help of the Rodriguez formula:

$$v_{mn}(x) = \frac{A_{mn} B_n}{\sigma^m(x)\rho(x)} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} \{\rho_n(x)\} \tag{7}$$

where

$$A_{mn} = (-1)^m \prod_{k=0}^{m-1} \mu_{kn} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left(-\frac{\lambda_{n+k}}{n+k}\right). \tag{8}$$

The leading coefficients of the orthogonal polynomial $y_n(x) = a_n x^n + b_n x^{n-1} + \cdots$ can be calculated [13]

$$a_n = B_n \prod_{k=0}^{n-1} \left(-\frac{\lambda_{n+k}}{n+k}\right) \tag{9}$$

hence it follows that $A_{nn} B_n = n! a_n$.

We address the construction of the raising and lowering operators for the generalized OPs using the Rodriguez formula, as we did in a recent work [1].

We have from equation (7)

$$\begin{aligned} v_{m,n+1}(x) &= \frac{A_{m,n+1} B_{n+1}}{\sigma^m \rho(x)} \frac{\mathrm{d}^{n+1-m}}{\mathrm{d}x^{n+1-m}} \{\rho_n(x)\} = \frac{A_{m,n+1} B_{n+1}}{\sigma^m \rho(x)} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} \{\tau_n(x)\rho_n(x)\} \\ &= \frac{A_{m,n+1} B_{n+1}}{\sigma^m(x)\rho(x)} \left\{ \tau_n(x) \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}} \{\rho_n(x)\} + (n-m)\tau_n' \frac{\mathrm{d}^{n-m-1}}{\mathrm{d}x^{n-m-1}} \{\rho_n(x)\} \right\} \\ &= \frac{B_{n+1}}{B_n} \left\{ \frac{A_{m,n+1}}{A_{mn}} \tau_n(x) v_{mn}(x) + (n-m) \frac{A_{m,n+1}}{A_{m+1,n}} \tau_n' \sigma(x) v_{mn}'(x) \right\} \\ &= \frac{B_{n+1}}{B_n} \left\{ \frac{n+1}{n-m+1} \frac{n}{\lambda_n} \frac{\lambda_{n+m}}{n+m} \tau_n(x) v_{mn}(x) - \frac{n+1}{n-m+1} \frac{n}{\lambda_n} \tau_n' \sigma(x) v_{mn}'(x) \right\}. \end{aligned} \tag{10}$$

The right-hand side can be considered the raising operator that, when applied to $v_{mn}(x)$, gives a new polynomial of higher order $v_{m,n+1}(x)$.

In order to evaluate the lowering operator we need a recurrence relation for the generalized polynomials. We write

$$x v_{mn}(x) = \sum_{k=0}^{n+1} c_{kn} v_{mk}(x) \qquad c_{kn} = \frac{1}{d_{mk}^2} \int_a^b v_{mk}(x) x v_{mn} \rho_m(x)\,\mathrm{d}x. \tag{11}$$

From the orthogonality condition (5) we deduce

$$\int_a^b v_{mn}(x) x^r \rho_m(x)\,\mathrm{d}x = 0 \qquad \text{for} \quad r < n-m.$$

Since $xp_k^{(m)}(x)$ is a polynomial of order $k+1-m$ it follows that $c_{kn}=0$ if $k+1-m<n-m$, or $k+1<n$. Hence

$$x v_{mn} = \tilde{\alpha}_n v_{m,n+1}(x) + \tilde{\beta}_n v_{mn}(x) + \tilde{\gamma}_n v_{m,n+1}(x) \tag{12}$$

where $\tilde{\alpha}_n = c_{n+1,n},\ \tilde{\beta}_n = c_{nm},\ \tilde{\gamma}_n = c_{n-1,n}$.

The coefficients $\tilde{\alpha}_n$, $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ can be expressed in terms of the squared norm d_n^2 and the leading coefficients a_n and b_n in $y_n(x)$.

From equation (11) it can be proven that $d_{mk}^2 c_{kn} = d_{mn}^2 c_{nk}$.

Since $\tilde{\alpha}_{n-1} = c_{n,n-1}$, $\tilde{\gamma}_n = c_{n-1,n}$, if we put $k=n-1$ we obtain

$$c_{n-1,n} d_{m,n-1}^2 = c_{n,n-1} d_{mn}^2$$

hence

$$\tilde{\gamma}_n = \tilde{\alpha}_{n-1} \frac{d_{m,n}^2}{d_{m,n-1}^2}.$$

Introducing the expansion $y_n(x) = a_n x^n + b_n x^{n-1} + \cdots$ in equation (12) and comparing the coefficients of the highest terms, we have

$$a_n(n-m+1) = \tilde{\alpha}_n a_{n+1}(n+1) \qquad b_n(n-m) = \tilde{\alpha}_n b_{n+1} n + \tilde{\beta}_n a_n n.$$

Hence

$$\tilde{\alpha}_n = \frac{a_n}{a_{n+1}} \frac{n-m+1}{n+1} \tag{13}$$

$$\tilde{\beta}_n = \frac{b_n}{a_n} \frac{(n-m)}{n} - \frac{b_{n+1}}{a_{n+1}} \frac{n+1-m}{n+1} \tag{14}$$

$$\tilde{\gamma}_n = \frac{a_{n-1}}{a_n} \frac{n-m}{n} \frac{d_{m,n}^2}{d_{m,n-1}^2}. \tag{15}$$

Substituting equation (9) in $\tilde{\alpha}_n$ we obtain

$$\tilde{\alpha}_n = \frac{-B_n}{B_{n+1}} \frac{n-m+1}{n+1} \frac{\lambda_n}{n} \frac{2n}{\lambda_{2n}} \frac{2n+1}{\lambda_{2n+1}}. \tag{16}$$

Hence equation (10) can be written

$$\tilde{\alpha}_n \frac{\lambda_{2n}}{2n} v_{m,n+1}(x) = \left\{ \frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau_n'} v_{mn}(x) - \sigma(x) v_{mn}'(x) \right\}. \tag{17}$$

Inserting equation (12) into equation (17) we obtain

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} v_{m,n-1}(x) = \left\{ -\frac{\lambda_{n+m}}{n+m} \frac{\tau_n(x)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(x - \tilde{\beta}_n) \right\} v_{mn}(x) + \sigma(x) v_{mn}'(x). \tag{18}$$

The right-hand sides of equations (17) and (18) can be considered the raising and lowering operators for the generalized classical OPs with respect to the index n.

All the constants $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n, \lambda_n, \tau_n'$ can be calculated from the tables of Nikiforov *et al* [12].

Now we define the orthonormalized function

$$\psi_{mn}(x) = d_{mn}^{-1} \sqrt{\rho_m(x)} v_{mn}(x) \tag{19}$$

hence

$$\psi_{mn}'(x) = \frac{1}{2} \frac{\rho_m'(x)}{\rho(x)} \psi_{mn}(x) + d_{mn}^{-1} \sqrt{\rho_m(x)} v_{mn}'(x) = \frac{1}{2} \frac{\tau_{m-1}(x)}{\sigma(x)} \psi_{mn}(x) + d_{mn}^{-1} \sqrt{\rho_m(x)} v_{mn}'(x). \tag{20}$$

Multiplying equation (17) by $d_{mn}^{-1}\sqrt{\rho_m(x)}$ and substituting equation (20) into equation (17) we obtain

$$\tilde{\alpha}_n \frac{\lambda_{2n}}{2n}\frac{d_{m,n+1}}{d_{mn}}\psi_{m,n+1}(x) = \frac{\lambda_{n+m}}{n+m}\frac{\tau_n(x)}{\tau_n'}\psi_{mn}(x) + \frac{1}{2}\tau_{m-1}(x)\psi_{mn}(x) - \sigma(x)\psi_{mn}'(x)$$
$$= L^{+}(x,n)\psi_{m,n}(x). \tag{21}$$

Similarly

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{2n}\frac{d_{m,n-1}}{d_{mn}}\psi_{m,n-1}(x) = \left\{-\frac{\lambda_{n+m}}{n+m}\frac{\tau_n(x)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(x-\tilde{\beta}_n) - \frac{1}{2}\tau_{m-1}(x)\right\}\psi_{mn}(x)$$
$$+\sigma(x)\psi_{mn}'(x) = L^{-}(x,n)\psi_{m,n}(x) \tag{22}$$

which can be considered the raising and lowering operators for the generalized orthonormal functions $\psi_{mn}(x)$. These operators are mutually adjoint with respect to the scalar product of unit weight.

Following the same procedure as in [1] we can factorize the raising and lowering operators as follows

$$L^{-}(x,n+1)L^{+}(x,n) = \mu(n) - \sigma(x)H(x,n)$$
$$L^{+}(x,n)L^{-}(x,n+1) = \mu(n) - \sigma(x)H(x,n+1)$$

where

$$\mu(n) = \frac{\lambda_{2n}}{2n}\frac{\lambda_{2n+2}}{2n+2}\tilde{\alpha}_n\tilde{\gamma}_{n+1}$$

and $H(x,n)$ is the differential operator derived from the left-hand side of equation (3) after substituting $\psi_{mn}(x)$ instead of $v_{mn}(x)$.

Notice that the factorization of the raising and lowering operators is defined in a basis independent manner, which is equivalent to the Hull–Infeld method.

3. Raising and lowering operators for generalized classical OPs of a discrete variable on a uniform lattice

Let $y_n(x)$ be an orthogonal polynomial of a discrete variable satisfying the difference equation [14]

$$\sigma(x)\Delta\nabla y_n(x) + \tau(x)\Delta y_n(x) + \lambda_n y_n(x) = 0 \tag{23}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degrees, respectively,

$$\lambda_n = -n\left(\tau' + \tfrac{1}{2}(n-1)\sigma''\right) \tag{24}$$

and the forward and backward difference operators are, respectively,

$$\Delta f(x) = f(x+1) - f(x) \qquad \nabla f(x) = f(x) - f(x-1).$$

It can be proven [15] that the differences of $y_n(x)$, namely $\Delta^m y_n(x) = v_{mn}(x)$, satisfy a similar equation of hypergeometric type

$$\sigma(x)\Delta\nabla v_{mn}(x) + \tau_m(x)\Delta v_{mn}(x) + \mu_{mn}v_{mn}(x) = 0 \tag{25}$$

with

$$\tau_m(x) = \tau(x+m) + \sigma(x+m) - \sigma(x)$$
$$\mu_{mn} = \lambda_n - \lambda_m = -(n-m)\left(\tau' + \frac{n+m-1}{2}\sigma''\right) \qquad m = 0, 1, \ldots, n-1.$$

We call the polynomials $v_{mn}(x)$ the generalized classical OPs of a discrete variable, and among these we find the Hahn, Chebyshev, Meixner, Kravchuk and Charlier polynomials.

The classical OPs of a discrete variable satisfy an orthogonality property with respect to the weight function $\rho(x)$

$$\sum_{x=a}^{b-1} y_\ell(x) y_n(x) \rho(x) = \delta_{\ell n} d_n^2. \tag{26}$$

Similarly, the generalized classical OPs of a discrete variable satisfy the orthogonality property

$$\sum_{x=a}^{b-1} v_{m\ell}(x) v_{mn}(x) \rho_m(x) = \delta_{\ell n} d_{mn}^2 \tag{27}$$

where d_n^2 and d_{mn}^2 are normalization constants. It can be proven that [16]

$$d_{mn}^2 = d_{nn}^2 \left(\prod_{k=m}^{n-1} \mu_{kn} \right)^{-1} \qquad d_{0n}^2 = d_{nn}^2 \left(\prod_{k=0}^{n-1} \mu_{kn} \right)^{-1}.$$

If we eliminate d_{nn} in the above equations we obtain

$$d_{mn}^2 = d_{0n}^2 \prod_{k=0}^{m-1} \mu_{kn}. \tag{28}$$

The generalized classical OPs of a discrete variable can be calculated from the weight function $\rho_m(x)$ with the formula [16]

$$v_{mn}(x) = \frac{A_{mn} B_n}{\rho_m(x)} \nabla^{n-m} \{\rho_n(x)\} \tag{29}$$

where

$$A_{mn} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left(-\frac{\lambda_{n+k}}{n+k} \right) \tag{30}$$

$$B_n = \frac{\Delta^n y_n(x)}{A_{nn}}. \tag{31}$$

The leading coefficients of the classical OPs of a discrete variable $y_n(x) = a_n x^n + b_n x^{n-1} + \cdots$ are given by [17]

$$a_n = B_n \prod_{k=0}^{n-1} \left(-\frac{\lambda_{n+k}}{n+k} \right) \tag{32}$$

from which it follows that $A_{nn} B_n = n! a_n$.

We now have all the necessary ingredients to construct the raising and lowering operators for the generalized OPs of a discrete variable in analogy with those of a continuous variable. From equation (29) we have

$$\begin{aligned} v_{m,n+1}(x) &= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \nabla^{n-m+1} \{\rho_{n+1}(x)\} \\ &= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \nabla^{n-m} \{\Delta \rho_{n+1}(x-1)\} \\ &= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \nabla^{n-m} \{\tau_n(x) \rho_n(x)\} \\ &= \frac{A_{m,n+1} B_{n+1}}{\rho_m(x)} \{\tau_n(x) \nabla^{n-m} \rho_n(x) + (n-m) \tau_n' \nabla^{n-m-1} \rho_n(x-1)\}. \end{aligned} \tag{33}$$

From equation (29) we have

$$\nabla^{n-m}\{\rho_n(x)\} = \frac{\rho_m(x)}{A_{mn}B_n} v_{mn}(x)$$

$$\nabla^{n-m-1}\{\rho_n(x-1)\} = \frac{\sigma(x)\rho_m(x)}{A_{m+1,n}B_n}\Delta^{m+1} y_n(x-1) = \frac{\sigma(x)\rho_m(x)}{A_{m+1,n}B_n} = \nabla v_{mn}(x).$$

Substituting the last two expressions into equation (33) and using equation (30) we obtain

$$v_{m,n+1}(x) = \frac{B_{n+1}}{B_n}\left\{\frac{n+1}{n+1-m}\frac{\lambda_{n+m}}{n+m}\frac{n}{\lambda_n}\tau_n(x)v_{mn}(x) - \frac{n+1}{n+1-m}\frac{n}{\lambda_n}\tau_n'\sigma(x)\nabla v_{mn}(x)\right\} \quad (34)$$

which raises in one step the order of the generalized polynomials in terms of the polynomials $v_{mn}(x)$ and $\nabla v_{mn}(x)$.

In order to evaluate the lowering operator we calculate a recurrence relation for the generalized OPs of a discrete variable. We write

$$x v_{mn}(x) = \sum_{k=0}^{n+1} c_{kn} v_{mk}(x) \qquad c_{kn} = \frac{1}{d_{mk}^2}\sum_{x=a}^{b-1} v_{mk}(x) x v_{mn}(x)\rho_m(x). \quad (35)$$

As in the case of the continuous variable $c_{kn} = 0$, if $k+1 < n$. Hence

$$x v_{mn}(x) = \tilde{\alpha}_n v_{m,n+1}(x) + \tilde{\beta}_n v_{mn}(x) + \tilde{\gamma}_n v_{m,n-1}(x) \quad (36)$$

where $\tilde{\alpha}_n = c_{n+1,n}$, $\tilde{\beta}_n = c_{nn}$, $\tilde{\gamma}_n = c_{n-1,n}$.

From equation (35) it follows that $d_{mk}^2 c_{kn} = d_{mn}^2 c_{nk}$.

Since $\tilde{\alpha}_{n-1} = c_{n,n-1}$, $\tilde{\gamma}_n = c_{n-1,n}$, if we put $k = n-1$, we obtain $c_{n-1,n}d_{m,n-1}^2 = c_{n,n-1}d_{m,n}^2$, hence

$$\tilde{\gamma}_n = \tilde{\alpha}_{n-1}\frac{d_{m,n}^2}{d_{m,n-1}^2}.$$

Introducing the expansion $y_n(x) = a_n x^n + b_n x^{n-1} + \cdots$ in equation (36), comparing the coefficients of the highest terms, and using

$$\Delta^m x^n = n(n-1)\cdots(n-m+1)x^{n-m} + \frac{m}{2}n(n-1)\cdots(n-m)x^{n-m+1} + \cdots$$

we obtain

$$a_n(n-m+1) = \tilde{\alpha}_n a_{n+1}(n+1)$$

$$a_n n(n-m)\frac{m}{2} + b_n(n-m) = \tilde{\alpha}_n b_{n+1} n + \tilde{\beta}_n a_n n + \tilde{\alpha}_n a_{n+1}(n+1)n\frac{m}{2}.$$

From these relations and from equation (32) we obtain

$$\tilde{\alpha}_n = \frac{a_n}{a_{n+1}}\frac{n-m+1}{n+1} = -\frac{B_n}{B_{n+1}}\frac{\lambda_n}{n}\frac{2n}{\lambda_{2n}}\frac{2n+1}{\lambda_{2n+1}}\frac{n-m+1}{n+1} \quad (37)$$

$$\tilde{\beta}_n = \frac{b_n}{a_n}\frac{(n-m)}{n} - \frac{b_{n+1}}{a_{n+1}}\frac{n-m+1}{n+1} - \frac{m}{2} \quad (38)$$

$$\tilde{\gamma}_n = \frac{a_{n-1}}{a_n}\frac{n-m}{n}\frac{d_{m,n}^2}{d_{m,n-1}^2}. \quad (39)$$

Hence equation (34) can be written

$$\tilde{\alpha}_n\frac{\lambda_{2n}}{2n} v_{m,n+1}(x) = \frac{\lambda_{n+m}}{n+m}\frac{\tau_n(x)}{\tau_n'} v_{mn}(x) - \sigma(x)\nabla v_{mn}(x). \quad (40)$$

Inserting equation (36) into equation (40) we obtain

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{2n} v_{m,n-1}(x) = -\frac{\lambda_{n+m}}{n+m}\frac{\tau_n(x)}{\tau_n'} v_{mn}(x) + \frac{\lambda_{2n}}{2n}(x - \tilde{\beta}_n) v_{mn}(x) + \sigma(x)\nabla v_{mn}(x). \tag{41}$$

The right-hand sides of equations (40) and (41) can be considered the raising and lowering operators with respect to the index n for the generalized OPs of a discrete variable on a homogeneous lattice.

All the constants $\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_n, \lambda_n, \tau_n'$ can be calculated from the tables of Nikiforov *et al* [18]. Obviously, when $m = 0, \tilde{\alpha}_n, \tilde{\beta}_n$ and $\tilde{\gamma}$ become α_n, β_n and γ_n, respectively.

Now we define the orthonormal function of a discrete variable

$$\phi_{mn}(x) = d_{mn}^{-1}\sqrt{\rho_m(x)} v_{mn}(x).$$

Using the identity $\frac{\nabla\rho_m(x)}{\rho_m(x)} = \frac{\tau_{m-1}(x)}{\tau_{m-1}(x)+\sigma(x)}$ and the properties of the backwards operator we obtain

$$\nabla\phi_{mn}(x) = \sqrt{\frac{\sigma(x)}{\tau_{m-1}(x)+\sigma(x)}} d_{mn}^{-1}\sqrt{\rho_m(x)}\nabla v_{mn}(x) + \frac{\tau_{m-1}(x)}{\sqrt{\sigma(x)}+\sqrt{\tau_{m-1}(x)+\sigma(x)}} \frac{\phi_{mn}(x)}{\sqrt{\sigma(x)+\tau_{m-1}(x)}}. \tag{42}$$

Multiplying both sides of equation (40) by $d_{mn}^{-1}\sqrt{\rho_m(x)}$ and inserting the value $d_{mn}^{-1}\sqrt{\rho_m(x)}$ $\nabla v_{mn}(x)$ obtained in equation (42), we obtain

$$\begin{aligned}\tilde{\alpha}_n \frac{\lambda_{2n}}{2n}\frac{d_{m,n+1}}{d_{m,n}}\phi_{m,n+1}(x) &= L^+(x,n)\phi_{mn}(x) \\ &= \left\{ +\frac{\lambda_{n+m}}{n+m}\frac{\tau_n(x)}{\tau_n'} + \frac{\sqrt{\sigma(x)}\tau_{m-1}(x)}{\sqrt{\sigma(x)}+\sqrt{\sigma(x)+\tau_{m-1}(x)}} \right\}\phi_{mn}(x) \\ &\quad - \sqrt{\sigma(x) - (\tau_{m-1}(x)+\sigma(x))}\nabla\phi_{mn}(x).\end{aligned} \tag{43}$$

Similarly

$$\begin{aligned}\tilde{\gamma}_n \frac{\lambda_{2n}}{2n}\frac{d_{m,n-1}}{d_{m,n}}\phi_{m,n-1}(x) &= L^-(x,n)\phi_{mn}(x) \\ &= \left\{ -\frac{\lambda_{n+m}}{n+m}\frac{\tau_n(x)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(x-\tilde{\beta}_n) - \frac{\sqrt{\sigma(x)}\tau_{m-1}(x)}{\sqrt{\sigma(x)}+\sqrt{\sigma(x)+\tau_{m-1}(x)}} \right\}\phi_{mn}(x) \\ &\quad + \sqrt{\sigma(x)\,(\tau_{m-1}(x)+\sigma(x))}\nabla\phi_{mn}(x).\end{aligned} \tag{44}$$

The expressions (43) and (44) can be considered the raising and lowering operators, respectively, for the generalized orthonormal functions on a homogeneous lattice. These operators are mutually adjoint with respect to the scalar product of unit weight.

Notice that in equations (43) and (44) the last term is proportional to $\nabla\phi_{mn}(s)$, which in the continuous limit becomes the derivative $\psi_{mn}'(x)$.

As in [1] we can factorize the raising and lowering operators as follows:

$$L^-(x,n+1)L^+(x,n) = \mu(n) + \mu(x+1,n)H(x,n)$$
$$L^+(x,n)L^-(x,n+1) = \mu(n) + \mu(x,n-1)H(x,n+1)$$

where

$$\mu(n) = \frac{\lambda_{2n}}{2n}\frac{\lambda_{2n+2}}{2n+2}\tilde{\alpha}_n\tilde{\gamma}_{n+1} \qquad \mu(x,n) = \frac{\lambda_n}{n}\frac{\tau_n(x)}{\tau_n'} - \sigma(x)$$

and $H(x,n)$ is the difference operator derived from the left-hand side of equation (25) after substituting $\phi_{mn}(x)$ instead of $v_{mn}(x)$.

4. Raising and lowering operators for classical OPs of a discrete variable on a non-uniform lattice

Let $y(s)$ be a function of a discrete variable satisfying the difference equation with respect to the lattice function $x(s)$

$$\sigma(s)\frac{\Delta}{\Delta x\,(s-1/2)}\left\{\frac{\nabla y(s)}{\Delta x(s)}\right\}+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0 \tag{45}$$

where $\sigma(s)\equiv\sigma\,[x(s)]$ and $\tau(s)\equiv\tau\,[x(s)]$ are functions of $x(s)$ of at most second and first degrees, respectively.

It can be proven [19] that the functions $v_k(s)$ connected with the solutions $y(s)$ by the relations

$$\begin{array}{ll} v_k(s)=\frac{\Delta v_{k-1}(s)}{\Delta x_{k-1}(s)} & v_0(s)=y(s) \\ x_k(s)=x\left(s+\frac{k}{2}\right) & k=0,1,2\ldots \end{array} \tag{46}$$

satisfy the difference equation

$$\sigma(s)\frac{\Delta}{\Delta x_k(s-1/2)}\left\{\frac{\nabla v_k(s)}{\nabla x_k(s)}\right\}+\tau_k(s)\frac{\Delta v_k(s)}{\Delta x_k(s)}+\mu_k v_k(s)=0 \tag{47}$$

where

$$\tau_k(s)=\frac{\sigma(s+k)-\sigma(s)+\tau(s+k)\Delta x\,(s+k-1/2)}{\Delta x\,(s+(k-1)/2)} \tag{48}$$

$$\mu_k=\lambda+\sum_{m=0}^{k-1}\frac{\Delta\tau_m(s)}{\Delta x_m(s)}=\lambda+\sum_{m=0}^{k-1}\tau'_m \tag{49}$$

provided the lattice functions $x(s)$ have the form

$$x(s)=c_1 s^2+c_2 s+c_3 \tag{50}$$

or

$$x(s)=c_1 q^s+c_2 q^{-s}+c_3 \tag{51}$$

with c_1, c_2, c_3 and q being arbitrary constants.

When $\mu_k=0$ for $k=n$ in equation (47) $v_n=\text{const}$. It can be proven that, when $k<n$, $v_k(s)$ is a polynomial in $x_k(s)$ and in particular for $k=0$, $v_0(s)=y(s)$ is a polynomial of degree n in $x(s)$ satisfying equation (45).

An explicit expression for λ_n, when $\mu_n=0$, is given by

$$\lambda_n=-\frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega}\left\{\operatorname{ch}(n-1)\omega\tau'+\frac{1}{2}\frac{\operatorname{sh}(n-1)\omega}{\operatorname{sh}\omega}\sigma''\right\} \tag{52}$$

where $\omega=\frac{1}{2}\ln q$, or $q=\mathrm{e}^{2\omega}$. For the square lattice (50) $\omega=0$, and for the q-lattice (51) we have

$$\frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega}=\frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}\equiv[n]_q\,.$$

The polynomial solutions of equation (45) satisfy the following orthogonality condition with respect to the weight functions $\rho(s)$, namely,

$$\sum_{s=a}^{b-1}y_\ell(s)y_n(s)\rho(s)\Delta x\left(s-\frac{1}{2}\right)=\delta_{\ell n}d_n^2. \tag{53}$$

Similarly for the differences of the polynomials $y_n(s)$, namely,

$$v_{mn}(s) \equiv \Delta^{(m)}\left[y_n(s)\right] = \Delta_{m-1}\Delta_{m-2}\dots\Delta_0\left[y_n(s)\right] \qquad \Delta_k \equiv \frac{\Delta}{\Delta x_k(s)}$$

it holds that

$$\sum_{s=a}^{b-k-1} v_{m\ell}(s)v_{mn}(s)\rho_m(s)\Delta x_m\left(s-\frac{1}{2}\right) = \delta_{\ell n}d_{mn}^2 \tag{54}$$

where $\rho_m(s) = \rho(s+m)\prod_{i=1}^{m}\sigma(s+i)$.

It can be proven that the normalization constants satisfy

$$d_{mn}^2 = d_{nn}^2\left(\prod_{k=m}^{n-1}\mu_{kn}\right)^{-1} \qquad d_{0n}^2 = d_{nn}^2\left(\prod_{k=0}^{n-1}\mu_{kn}\right)^{-1}$$

from which d_{nn}^2 can be eliminated:

$$d_{mn}^2 = d_{0n}^2\prod_{k=0}^{m-1}\mu_{kn}. \tag{55}$$

A particular solution of equation (45) when $\lambda = \lambda_n$ is given by the Rodriguez-type formula

$$y_n(s) = \frac{B_n}{\rho(s)}\nabla_n^{(n)}\left[\rho_n(s)\right] = \frac{B_n}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\cdots\frac{\nabla}{\nabla x_n(s)}\left[\rho_n(s)\right]. \tag{56}$$

A solution of equation (47), when μ_k is restricted to λ_n, namely, $\mu_{mn} = \mu_m(\lambda_n) = \lambda_n - \lambda_m, 0, 1, \dots, n-1$, is given by

$$v_{mn}(s) = \frac{A_{mn}B_n}{\rho_m(s)}\nabla_n^{(n-m)}\left[\rho_n(s)\right] = \frac{A_{mn}B_n}{\rho_m(s)}\frac{\nabla}{\nabla x_{m+1}(s)}\cdots\frac{\nabla}{\nabla x_{n-1}(s)}\frac{\nabla}{\nabla x_n(s)}\left[\rho_n(s)\right] \tag{57}$$

where

$$A_{mn} = (-1)^m\prod_{k=0}^{m-1}\mu_{kn} = \frac{[n]!}{[n-m]!}\prod_{k=0}^{m-1}\frac{\lambda_{n+k}}{[n+k]} \qquad B_n = A_{nn}^{-1}\Delta^{(n)}y_n(s). \tag{58}$$

Formulae (56) and (57) can be written in terms of the mean difference operator [4] $\delta f(s) = f\left(s+\frac{1}{2}\right) - f\left(s-\frac{1}{2}\right) = \Delta f\left(s-\frac{1}{2}\right) = \nabla f\left(s+\frac{1}{2}\right)$, that is to say,

$$y_n(s) = \frac{B_n}{\rho(s)}\left[\frac{\delta}{\delta x(s)}\right]^n\rho_n\left(s-\frac{n}{2}\right) \tag{59}$$

$$v_{mn}(s) = \frac{A_{mn}B_n}{\rho_m(s)}\left[\frac{\delta}{\delta x\left(s+\frac{m}{2}\right)}\right]^{n-m}\rho_n\left(s-\frac{n}{2}+\frac{m}{2}\right). \tag{60}$$

In order to obtain the raising and lowering operators of the classical OPs on a non-homogeneous lattice, we apply the Rodriguez formula (56)

$$y_{n+1}(s) = \frac{B_{n+1}}{\rho(s)}\nabla_{n+1}^{(n+1)}\left\{\rho_{n+1}(s)\right\} = \frac{B_{n+1}}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\cdots\frac{\nabla}{\nabla x_{n+1}(s)}\left\{\rho_{n+1}(s)\right\}.$$

Since

$$\frac{\nabla\rho_{n+1}(s)}{\nabla x_{n+1}(s)} = \frac{\Delta\rho_{n+1}(s-1)}{\Delta x_{n+1}(s-1)} = \frac{\Delta\left\{\sigma(s)\rho_n(s)\right\}}{\Delta x_n\left(s-\frac{1}{2}\right)} = \tau_n(s)\rho_n(s)$$

using equation (59) we have

$$y_{n+1}(s) = \frac{B_{n+1}}{\rho(s)} \nabla_n^{(n)} \{\tau_n(s)\rho_n(s)\} = \frac{B_{n+1}}{\rho(s)} \left[\frac{\delta}{\delta x(s)}\right]^n \left\{\tau_n\left(s-\frac{n}{2}\right)\rho_n\left(s-\frac{n}{2}\right)\right\}$$
$$= \frac{B_{n+1}}{\rho(s)} \left\{\tau_n(s)\left[\frac{\delta}{\delta x(s)}\right]^n \rho_n\left(s-\frac{n}{2}\right) + \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega}\tau_n' \left[\frac{\delta}{\delta x\left(s-\frac{1}{2}\right)}\right]^{n-1} \rho_n\left(s-\frac{n}{2}-\frac{1}{2}\right)\right\}. \tag{61}$$

The last step can be proven by induction for both cases of $x(s)$ on a non-homogeneous lattice (50) and (51).

First of all, we transform the properties of these functions [5] given by

$$x(s+n) - x(s) = \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega} \nabla x\left(s+\frac{n+1}{2}\right) \qquad x(s+n) + x(s) = \operatorname{ch} n\omega\, x(s) + \text{const}$$

into the difference relations

$$\delta x\left(s+\frac{n}{2}\right) - \delta x\left(s-\frac{n}{2}\right) = \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega}\left\{\delta x\left(s+\frac{1}{2}\right) - \delta x\left(s-\frac{1}{2}\right)\right\} \tag{62}$$

$$\frac{1}{2}\left\{\delta x\left(s+\frac{n}{2}\right) + \delta x\left(s-\frac{n}{2}\right)\right\} = \operatorname{ch} n\omega\, \delta x(s). \tag{63}$$

Suppose it is true that for any two functions of a discrete variable it holds that

$$\left(\frac{\delta}{\delta x(s)}\right)^n \{f(s)g(s)\} = f\left(s+\frac{n}{2}\right)\left(\frac{\delta}{\delta x(s)}\right)^n g(s)$$
$$+ \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega} \frac{\delta f\left(s+\frac{n-1}{2}\right)}{\delta x\left(s+\frac{n-1}{2}\right)} \left(\frac{\delta}{\delta x\left(s-\frac{1}{2}\right)}\right)^{n-1} g\left(s-\frac{1}{2}\right) + \cdots.$$

Then using the properties of the mean operator we have

$$\left(\frac{\delta}{\delta x(s)}\right)^{n+1} \{f(s)g(s)\} = f\left(s+\frac{n+1}{2}\right)\left(\frac{\delta}{\delta x(s)}\right)^{n+1} g(s)$$
$$+ \frac{\delta f\left(s+\frac{n}{2}\right)}{\delta x(s)} \left(\frac{\delta}{\delta x\left(s-\frac{1}{2}\right)}\right)^{n} g\left(s-\frac{1}{2}\right)$$
$$+ \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega} \frac{\delta f\left(s+\frac{n}{2}\right)}{\delta x\left(s+\frac{n}{2}\right)} \frac{\delta}{\delta x(s)} \left(\frac{\delta}{\delta x\left(s-\frac{1}{2}\right)}\right)^{n-1} g\left(s-\frac{1}{2}\right) + \cdots.$$

The second and third terms on the right-hand side can be written

$$\frac{\delta f\left(s+\frac{n}{2}\right)}{\delta x\left(s+\frac{n}{2}\right)} \left\{\frac{\delta x\left(s+\frac{n}{2}\right)}{\delta x(s)} + \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega}\frac{\delta x\left(s-\frac{1}{n}\right)}{\delta x(s)}\right\} \left(\frac{\delta}{\delta x\left(s-\frac{1}{2}\right)}\right)^{n} g\left(s-\frac{1}{2}\right).$$

Using equations (62) and (63) the expression between curly brackets is equal to $\operatorname{sh}(n+1)\omega/\operatorname{sh}\omega$, therefore

$$\left(\frac{\delta}{\delta x(s)}\right)^{n+1} \{f(s)g(s)\} = f\left(s+\frac{n+1}{2}\right)\left(\frac{\delta}{\delta x(s)}\right)^{n+1} g(s)$$
$$+ \frac{\operatorname{sh}(n+1)\omega}{\operatorname{sh}\omega} \frac{\delta f\left(s+\frac{n}{2}\right)}{\delta x\left(s+\frac{n}{2}\right)} \left(\frac{\delta}{\delta x\left(s-\frac{1}{2}\right)}\right)^{n} g\left(s-\frac{1}{2}\right) + \cdots$$

as required. Substituting $f(x) = \tau_n\left(s - \frac{n}{2}\right)$ and $g(s) = \rho_n\left(s - \frac{n}{2}\right)$, the terms of lower degree become zero, due to the properties of function $\tau_n(s)$. Therefore equation (61) is proven.

Using equation (60) for $m = 1$ we have

$$\frac{\nabla y_n(s)}{\nabla x(s)} = \frac{\Delta y_n(s-1)}{\Delta x(s-1)} = v_{1n}(s-1) = \frac{A_{1n} B_n}{\rho_1(s-1)} \left(\frac{\delta}{\delta x\left(s - \frac{1}{2}\right)}\right)^{n-1} \rho_n\left(s - \frac{n}{2} - \frac{1}{2}\right).$$

Therefore, equation (61) can be written

$$y_{n+1}(s) = \frac{B_{n+1}}{B_n}\left\{\tau_n(s) y_n(s) + \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega} \frac{\tau_n'}{A_{1n}} \sigma(s) \frac{\nabla y_n(s)}{\nabla x(s)}\right\}. \qquad (64)$$

Alvarez-Nodarse and Costas-Santos [6] and Alvarez-Nodarse and Arvesú [7] have given the same formula for the lattice (51). Here we have proven a similar expression for both cases (50) and (51).

From equation (64) we can calculate the raising and lowering operators. Instead, we proceed to the general case in section 5, and then take the value $m = 0$.

5. Raising and lowering operators for generalized classical OPs of a discrete variable on a non-uniform lattice

From equations (57) and (60) we obtain

$$\begin{aligned}
v_{m,n+1}(s) &= \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \nabla_{n+1}^{(n+1-m)} \{\rho_{n+1}(s)\} \\
&= \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \nabla_{n}^{(n-m)} \{\tau_n(s)\rho_n(s)\} \\
&= \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} \left(\frac{\delta}{\delta x\left(s + \frac{m}{2}\right)}\right)^{n-m} \left\{\tau_n\left(s - \frac{n-m}{2}\right)\rho_n\left(s - \frac{n-m}{2}\right)\right\} \\
&= \frac{A_{m,n+1} B_{n+1}}{\rho_m\left(s' - \frac{m}{2}\right)} \left(\frac{\delta}{\delta x(s')}\right)^{n-m} \left\{\tau_n\left(s' - \frac{n}{2}\right)\rho_n\left(s' - \frac{n}{2}\right)\right\}.
\end{aligned}$$

With respect to the new variable $s' = s + \frac{m}{2}$, this expression can be easily differentiated as in equation (61) giving

$$\begin{aligned}
v_{m,n+1}(s) = \frac{A_{m,n+1} B_{n+1}}{\rho_m(s)} &\left\{\tau_n(s) \left(\frac{\delta}{\delta x\left(s + \frac{m}{2}\right)}\right)^{n-m} \rho_n\left(s - \frac{n-m}{2}\right)\right. \\
&\left. + \frac{\operatorname{sh}(n-m)\omega}{\operatorname{sh}\omega} \tau_n' \left(\frac{\delta}{\delta x\left(s + \frac{m-1}{2}\right)}\right)^{n-m-1} \rho_n\left(s - \frac{n}{2} + \frac{m-1}{2}\right)\right\}.
\end{aligned}$$

From equation (60) we obtain

$$\begin{aligned}
\frac{\nabla v_{m,n}(s)}{\nabla x(s)} &= \frac{\Delta v_{m,n}(s-1)}{\Delta x(s-1)} = v_{m+1,n}(s-1) \\
&= \frac{A_{m+1,n} B_n}{\rho_{m+1}(s-1)} \left(\frac{\delta}{\delta x\left(s + \frac{m-1}{2}\right)}\right)^{n-m-1} \rho_n\left(s - \frac{n}{2} + \frac{m-1}{2}\right)
\end{aligned}$$

Using this result and the values for $A_{m,n}$ given in equation (58) we obtain the raising operator for $v_{mn}(s)$, namely,

$$v_{m,n+1}(s) = \frac{B_{n+1}}{B_n}\left\{\frac{[n+1]}{[n+1-m]} \frac{\lambda_{n+m}}{[n+m]} \frac{[n]}{\lambda_n} \tau_n(x) v_{mn}(x) - \frac{[n+1]}{[n+1-m]} \frac{[n]}{\lambda_n} \tau_n' \sigma(x) \frac{\nabla v_{m,n}(s)}{\nabla x(s)}\right\} \qquad (65)$$

with $[n] \equiv \frac{\operatorname{sh} n\omega}{\operatorname{sh}\omega}$ corresponding to all values of lattice functions $x(s)$ given in equations (50) and (51).

In order to construct the lowering operator we use the recurrence relation

$$x_m(s)v_{mn}(s) = \tilde{\alpha}_n v_{m,n+1}(s) + \tilde{\beta}_n v_{mn}(s) + \tilde{\gamma}_n v_{m,n-1}(s) \tag{66}$$

where $x_m(s) = x\left(s + \frac{m}{2}\right)$ and $v_{mn}(s) \equiv \Delta^{(m)} y_n(s)$.

We introduce the expansion $y_n(s) = a_n x^n(s) + b_n x^{n-1}(s) + \cdots$ in the recurrence relation (66). We have two cases [20]

(a) *Quadratic lattice:* $x(s) = s(s+1)$.

$$\Delta^{(m)} x^n(s) = n(n-1)\cdots(n-m+1)x_m^{n-m}(s) + \frac{1}{12^m} n(n-1)\cdots(n-m)(2n-2m+1)x_m^{n-m-1}(s)$$

which after substitution in the recurrence relation (66) gives

$$\tilde{\alpha}_n = \frac{a_n}{a_{n+1}} \frac{n-m+1}{n+1} \tag{67}$$

$$\tilde{\beta}_n = \frac{b_n}{a_n}\frac{n-m}{n} - \frac{b_{n+1}}{a_{n+1}}\frac{n-m+1}{n+1} - \frac{3}{12^m} \tag{68}$$

$$\tilde{\gamma}_n = \frac{a_{n-1}}{a_n}\frac{n-m}{n}\frac{d_{mn}^2}{d_{m,n-1}^2}. \tag{69}$$

(b) *Exponential lattice:* $x(s) = Aq^s + Bq^{-s}$.

$$\Delta^{(m)} x^n(s) = [n][n-1]\cdots[n-m+1]x_m^{n-m-1}(s) + Cx_m^{n-m-3}(s) + \cdots$$

which after substitution in the recurrence relation (66) gives

$$\tilde{\alpha}_n = \frac{a_n}{a_{n+1}}\frac{[n-m+1]}{[n+1]} \tag{70}$$

$$\tilde{\beta}_n = \frac{b_n}{a_n}\frac{[n-m]}{[n]} - \frac{b_{n+1}}{a_{n+1}}\frac{[n-m+1]}{[n+1]} \tag{71}$$

$$\tilde{\gamma}_n = \frac{a_{n-1}}{a_n}\frac{[n-m]}{[n]}\frac{d_{mn}^2}{d_{m,n-1}^2}. \tag{72}$$

Since $a_n = B_n \prod_{k=0}^{n-1}\left(-\frac{\lambda_{n+k}}{[n+k]}\right)$ we obtain

$$\tilde{\alpha}_n = -\frac{B_n}{B_{n+1}}\frac{\lambda_n}{[n]}\frac{[2n]}{\lambda_{2n}}\frac{[2n+1]}{\lambda_{2n+1}}\frac{[n-m+1]}{[n+1]} \tag{73}$$

which after substituting in equation (65) gives

$$\tilde{\alpha}_n \frac{\lambda_{2n}}{[2n]} v_{m,n+1}(s) = \frac{\lambda_{n+m}}{[n+m]}\frac{\tau_n(s)}{\tau_n'} v_{mn}(s) - \sigma(s)\frac{\nabla v_{mn}(s)}{\nabla x(s)}. \tag{74}$$

Inserting the recurrence relation (66) into equation (74) we obtain

$$\tilde{\gamma}_n \frac{\lambda_{2n}}{[2n]} v_{m,n-1}(s) = \left\{-\frac{\lambda_{n+m}}{[n+m]}\frac{\tau_n(s)}{\tau_n'} + \frac{\lambda_{2n}}{[2n]}\right\} v_{mn}(s) + \sigma(s)\frac{\nabla v_{mn}(s)}{\nabla x(s)}. \tag{75}$$

The last two equations can be considered the raising and lowering operators of generalized OPs on non-uniform lattices for the functions (50) and (51). In the first case the parameter $[n]$ should be taken as n.

In order to complete the picture, we define an orthonormal function

$$\Omega_{mn}(s) = d_{mn}^{-1}\sqrt{\rho_m(s)}v_{mn}(s). \tag{76}$$

Using the properties of the difference operator and the identity

$$\frac{\nabla\rho_m(s)}{\rho_m(s)} = \frac{\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}{\sigma(s)+\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)} \tag{77}$$

we obtain

$$\begin{aligned}\nabla\Omega_{mn}(s) = {} & \sqrt{\frac{\sigma(s)}{\sigma(s)+\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}}\,d_{mn}^{-1}\sqrt{\rho_m(s)}\nabla v_{mn}(s)\\ & + \frac{1}{\sqrt{\sigma(s)}+\sqrt{\sigma(s)+\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}}\\ & \times \frac{\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}{\sqrt{\sigma(s)+\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}}\Omega_{mn}(s).\end{aligned} \tag{78}$$

Multiplying both sides of equation (74) by $d_{mn}^{-1}\sqrt{\rho_m(s)}$ and substituting the value $d_{mn}^{-1}\sqrt{\rho_m(s)}\,\nabla v_{mn}(s)$ obtained in equation (78) we obtain

$$\begin{aligned}\tilde{\alpha}_n\frac{\lambda_{2n}}{2n}\frac{d_{m,n+1}}{d_{mn}}\Omega_{m,n+1}(s) &= L^+(s,n)\Omega_{mn}(s)\\ &= \left\{\frac{\lambda_{n+m}}{[n+m]}\frac{\tau_n(s)}{\tau_n'(s)} + \frac{\sqrt{\sigma(s)}\tau_{m-1}(s)}{\sqrt{\sigma(s)}+\sqrt{\sigma(s)+\tau_{m-1}(s)\Delta x_m\left(s-\frac{1}{2}\right)}}\frac{\nabla x_m\left(s+\frac{1}{2}\right)}{\nabla x(s)}\right\}\\ &\quad\times\Omega_{mn}(s) - \sqrt{\sigma(s)\sigma(s)+\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}\frac{\nabla\Omega_{mn}(s)}{\nabla x(s)}.\end{aligned} \tag{79}$$

Similarly

$$\begin{aligned}\tilde{\gamma}_n\frac{\lambda_{2n}}{2n}\frac{d_{m,n-1}}{d_{mn}}\Omega_{m,n-1}(s) &= L^-(s,n)\Omega_{mn}(s)\\ &= \left\{-\frac{\lambda_{n+m}}{[n+m]}\frac{\tau_n(s)}{\tau_n'} + \frac{\lambda_{2n}}{2n}(s-\tilde{\beta}_n) - \frac{\sqrt{\sigma(s)}\tau_{m-1}(s)}{\sqrt{\sigma(s)+\tau_{m-1}(s)\Delta x_m\left(s-\frac{1}{2}\right)}}\right.\\ &\quad\left.\times\frac{\nabla x_m\left(s+\frac{1}{2}\right)}{\nabla x(s)}\right\}\Omega_{mn}(s) + \sqrt{\sigma(s)\sigma(s)+\tau_{m-1}(s)\Delta x_{m-1}\left(s-\frac{1}{2}\right)}\frac{\nabla\Omega_{mn}(s)}{\nabla x(s)}.\end{aligned} \tag{80}$$

The last two expressions can be considered the raising and lowering operators for the generalized orthonormal functions on non-homogeneous lattices of the type (50) and (51). It can be proven that these operators are mutually adjoint with respect to the scalar product of unit weight.

As in the previous sections we can factorize the raising and lowering operators as follows:

$$\begin{aligned}L^-(s,n+1)L^+(s,n) &= \mu(n) + u(s+1,n)H(s,n)\\ L^+(s,n)L^-(s,n+1) &= \mu(n) + u(s,n-1)H(s,n+1)\end{aligned}$$

where

$$\mu(n) = \frac{\lambda_{2n}}{[2n]} \frac{\lambda_{2n+2}}{[2n+2]} \tilde{\alpha}_n \tilde{\gamma}_{n+1} \qquad u(s,n) = \frac{\lambda_n}{[n]} \frac{\tau_n(s)}{\tau_n'} - \frac{\sigma(s)}{\nabla x(s)}$$

and $H(s,n)$ is the difference operator derived from the left-hand side of equation (47) after substituting $\Omega_{mn}(s)$ instead of $v_{mn}(s)$ given in equation (76). Notice that the expressions for the factorization of the raising and lowering operators become the same expressions (32) and (33) given in [6].

6. Conclusions

We have developed the construction of raising and lowering operators for classical OPs of a discrete variable on a non-homogeneous lattice, extended also to the generalized OPs on homogeneous and non-homogeneous lattices.

In the last case (generalized OPs) the raising and lowering operators can be defined with respect to the index n, the order of the OP, or with respect to the index m, the order of the difference derivative of the generalized OP, or both [25].

In this paper, we have taken into account only the index n, although we have suggested how to complement the calculus with the index m. We have also introduced the orthornomal functions of unit weight, more suitable to quantum mechanical applications.

Our presentation leads to an easier method for the continuous limit (compare with a different presentation in [1]).

We have already worked out some physical applications of raising and lowering operators on homogeneous and non-homogeneous lattices. For instance, the quantum mechanical models for the harmonic oscillator in one dimension (Kravchuk OP), the hydrogen atom (generalized Meixner OP) [21], the Heisenberg equation of motion on the lattice (Hahn OP) [22] and the Dirac and Klein–Gordon equations on a homogeneous lattice (discrete exponential function) [23, 24].

Finally, studies of the connection between OPs on a non-homogeneous lattice and the $3nj$-Wigner coefficients and its application to spin network models in quantum gravity are now in progress.

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